Available online at www.sciencedirect.com
SCIENCE DIRECT
Topology and its Applications 149 (2005) 57–83
Topology and its Applications
www.elsevier.com/locate/topol

Measure extension theorems for T_0-spaces

Klaus Keimel [a], Jimmie D. Lawson [b,*,1]

[a] FB Mathematik, AG 14, Technical University, Darmstadt, Schlossgartenstr. 7, D-64289 Darmstadt, Germany
[b] Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA

Received 17 October 2003; accepted 12 February 2004

Abstract

The theme of this paper is the extension of continuous valuations on the lattice of open sets of a T_0-space to Borel measures. A general extension principle is derived that provides a unified approach to a variety of extension theorems including valuations that are directed suprema of simple valuations, continuous valuations on locally compact sober spaces, and regular valuations on coherent sober spaces.

MSC: 28C15; 06B35

Keywords: Valuation; Borel measure; Sober space

Introduction

This paper may be seen as a contribution to topological measure theory. While topological measure theory has been mainly concerned with Hausdorff spaces, we consider non-Hausdorff spaces. One major impetus for such considerations arises in the theory of continuous domains (see, for example, [1] or [8]) and their applications to denotational semantics or theoretical computation. In this setting one typically works with topological spaces that

* Corresponding author.
E-mail addresses: keimel@mathematik.tu-darmstadt.de (K. Keimel), lawson@math.lsu.edu (J.D. Lawson).
1 The second author was supported by a research grant of the Alexander von Humboldt-Stiftung during the work on the first six sections of the present paper, the first author was supported by a DFG grant during the elaboration of Section 7.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2004.02.019

satisfy the T_0-axiom but are far from being Hausdorff (the so-called "Scott topology" of the domain). The extended nonnegative reals $[0, +\infty]$ with the topology for which the upwards unbounded open intervals $]r, +\infty]$, $r \geqslant 0$, are the only open sets may serve as a typical example. The open sets in these semantic domains represent observable properties of programs. In the presence of probabilistic features, a certain property will be observed with a certain probability. For modeling probabilistic phenomena it seems therefore natural to consider set functions λ that assign a nonnegative real number $\lambda(U)$ to every open subset U of the semantic domain in such a way that some natural requirements are satisfied. These are (i) $\lambda(\emptyset) = 0$, (ii) whenever U, V are open sets with $U \subseteq V$, then $\lambda(U) \leqslant \lambda(V)$, (iii) whenever U, V are any open sets, then $\lambda(U) + \lambda(V) = \lambda(U \cup V) + \lambda(U \cap V)$. The modular law (iii) (or inclusion-exclusion principle) replaces the finite additivity that one requires for probabilities and measures on algebras of sets. But finite additivity for open sets would be too weak in the context of semantic domains, as it may happen there that open sets are never disjoint. A set function satisfying these three properties is called a *valuation*. Having probabilities in mind it would be natural to require the whole space to have value 1. In this paper we do not want to make this restriction. As for measures, we even allow the value $+\infty$.

Of course, one will wonder about the relation between valuations and Borel measures, that is, measures defined on the σ-algebra generated by the open sets. Clearly, every Borel measure restricts to a valuation on the open sets. Thus one will ask the

Question. Can one extend a given valuation to a Borel measure?

Of course, the answer will be NO in general. For a positive solution, one has to impose conditions on the spaces and some kind of continuity condition on the valuation. There have been several papers dealing with solutions to the extension problem in various situations (e.g., [2–4,11,14,9,13,18]).

In this paper we present a unified treatment of these extension theorems based on Theorem 2.4, a slight generalization of an old result due to Topsøe [17, Theorem 6.1], which is not easily accessible. This theorem allows us to derive the extension results due to Alvarez-Manilla et al. [4] for suprema of directed families of simple valuations (see Theorem 4.4), due to Alvarez-Manilla [2,3] for continuous valuations on locally compact sober spaces (see Theorem 5.3), due to Norberg and Vervaat [14] for certain valuations defined on the compact saturated subsets of a coherent sober space (see Theorem 6.5), and finally those due to Lawson [11] and Weidner [18,10] for continuous valuations on stably locally compact spaces (see Theorem 8.3). Our main goal is not novelty, but the presentation of a unified and essentially self-contained treatment (except for a rather standard extension theorem cited from Billingsley) of most of the important extension theorems pertaining to continuous valuations on general topological spaces. One perhaps new result is Theorem 6.8. Section 7 develops a theory of Radon measures for coherent sober spaces, improving the results of Section 6. The appropriate definition of a Radon measure is new as is the main result Theorem 7.3.

The classical methods of topological measure theory as developed in the Hausdorff setting cannot be generalized in a straightforward way. What are the main difficulties that one encounters in this more general situation and what new paths does one take?

Firstly, the classical extension theorems as well as ours depend on a compactness argument. A collection $\mathcal{K}$ of subsets of a set Ω is called (*semi-*)*compact* if every (countable) subfamily $(K_j)_{j\in\mathbb{N}}$ with the finite intersection property has a nonempty intersection. It can then be proved that the collection of all finite unions of members of $\mathcal{K}$ is also (semi-) compact (see, e.g., [12, Lemma I-6-1], [7, Section III-1]). In Hausdorff spaces, the collection of compact subsets is compact in this sense. In non-Hausdorff spaces this is no longer true. The reason is that compact sets need not be closed and that the intersection of finitely many compact sets need not be compact. In order to apply the compactness arguments one has to weaken the requirements: A collection $\mathcal{K}$ of subsets is called *monotone compact*, or *monocompact* for short, if every decreasing sequence of nonempty members of $\mathcal{K}$ has a nonempty intersection. For this weaker notion it is no longer true that the finite unions of members form a monotone compact collection, but one can show that the family of finite disjoint unions of members of $\mathcal{K}$ ordered by refinement is also monotone compact. This is the first key idea for our results, as there are the suitable collections of compact sets that turn out to be monotone compact (see Section 2).

Secondly open sets are in a sense too large in order to apply Carathéodory's method of constructing a measure through the outer measure derived from a valuation on the open sets. One has to replace the open sets by *crescents* which, by definition, are relative complements $U \setminus V$ of open sets U, V (see Section 3).

Thirdly, the appropriate choice of compact sets for applying the method of inner measures are those compact sets that we call *lenses*. These are obtained as relative complements $Q \setminus U$ of a *saturated* compact set Q and an open set U, where a compact set is called saturated if it is representable as an intersection of open sets (see Section 5).

Detailed expositions on semantic domains and the corresponding topological spaces can be found in [1,8].

1. Background from measure theory

A *ring* $\mathcal{A}$ of subsets of a set Ω is a collection of subsets containing the empty set and closed under finite unions, finite intersections and relative complements. A *semiring* is a collection $\mathcal{A}$ containing the empty set, closed under finite intersections, and having the property that the relative complement of two members of $\mathcal{A}$ can be written as a finite disjoint union of members of $\mathcal{A}$. A ring (respectively semiring) that contains Ω as a member is called an *algebra* (respectively *semialgebra*). (Algebras are often referred to alternatively as *fields*.) It is a standard and elementary result that the smallest ring containing a semiring consists of those sets that are finite disjoint unions of members of the semiring.

A *finitely additive measure* on $\mathcal{A}$ (for any of the preceding cases) is a monotone function $\lambda : \mathcal{A} \to [0, \infty]$ satisfying $\lambda(\emptyset) = 0$ and $\lambda(A) = \sum_{i=1}^{n} \lambda(A_i)$, whenever A and all A_i are in $\mathcal{A}$ and A is the disjoint union of the A_i. The last condition is called *finite additivity*; *countable additivity* is defined in an analogous way. A σ*-ring* or σ*-algebra* is one that additionally is closed under taking countable unions (and hence also countable intersections). A *measure* is a countably additive measure defined on a σ-algebra of sets.

A *lattice* $\mathcal{L}$ of subsets of Ω is a collection of subsets containing the empty set and closed under finite unions and finite intersections. A *valuation* on $\mathcal{L}$ is a function $\lambda : \mathcal{L} \to [0, \infty]$

satisfying (i) $\lambda(\emptyset) = 0$, (ii) $A \subseteq B$ in $\mathcal{L}$ implies $\lambda(A) \leqslant \lambda(B)$, and (iii) for all $A, B \in \mathcal{L}$, $\lambda(A) + \lambda(B) = \lambda(A \cup B) + \lambda(A \cap B)$. If properties (i), (ii), (iii) are satisfied, the set function λ is said to be *strict*, *monotone*, and *modular*, respectively. A basic result is the Smiley–Horn–Tarski theorem, which asserts that a valuation on a lattice $\mathcal{L}$ extends to a finitely additive measure on the smallest algebra $\mathcal{AL}$ containing $\mathcal{L}$; furthermore, if the valuation takes only finite values, then the extension is unique on the smallest ring of sets containing $\mathcal{L}$ (see, for example, [8, Chapter IV-9]). The ring of sets generated by a lattice is known to consist of all finite disjoint unions of sets of the form $A \setminus B$, where $A, B \in \mathcal{L}$. The relative complements $A \setminus B$, where $A, B \in \mathcal{L}$, form a semiring $\mathcal{SL}$ containing $\mathcal{L}$, and on this semiring the extension of λ is uniquely defined by $\lambda(A \setminus B) = \lambda(A) - \lambda(B \cap A)$, whenever $\lambda(A) < +\infty$. (If there are infinite values the extension may not be unique.)

The next theorem is a variant of the well-known Carathéodory Extension theorem. It gives a standard condition (usually stated for algebras or rings of sets) for extending a finitely additive measure on a semiring to a countably additive one on a σ-algebra (see, for example, [6, Theorems 11.1 and 10.3] for the semiring version).

Theorem 1.1. *If a finitely additive measure λ on a semiring $\mathcal{S}$ of subsets of a set Ω is countably additive on $\mathcal{S}$, then λ can be extended to a measure on a σ-algebra containing the semiring $\mathcal{S}$. This extension is unique on the smallest σ-ring of sets containing the members of $\mathcal{S}$ of finite measure.*

A finitely additive measure that satisfies the conditions of the next corollary is sometimes called *σ-smooth at $\emptyset$*.

Corollary 1.2. *Suppose that λ is a finite, finitely additive measure on a ring $\mathcal{A}$ and satisfies $\lim_n \lambda(A_n) = 0$ whenever A_n is a decreasing sequence in $\mathcal{A}$ such that $\bigcap_{n=1}^{\infty} A_n = \emptyset$. Then λ is countably additive on $\mathcal{A}$, and hence extends to a measure on the σ-algebra generated by $\mathcal{A}$.*

Proof. Consider a countable disjoint sequence (A_n) in $\mathcal{A}$ such that $A = \bigcup_n A_n \in \mathcal{A}$. The tails $T_n := A \setminus \bigcup_{i=1}^{n} A_i$ then form a decreasing sequence in $\mathcal{A}$ with empty intersection. It follows that $\lim_n \lambda(T_n) = 0$, equivalently $\lambda(A) - \lim_n \lambda(\bigcup_{i=1}^{n} A_i) = 0$, and hence that λ is countably additive on $\mathcal{A}$. The result now follows from the previous theorem. □

2. Extending finitely additive measures

In this section we give a mild generalization of an extension theorem of Topsøe [17, Theorem 1, Section 6]. We begin with our basic terminology and notation.

Let Ω be a set. A *paving* of Ω is simply a collection of subsets. A *partitioned subset* $\mathbf{P}$ consists of a partition of some $P \subseteq \Omega$. We call P the *carrier* of the partition. (We allow $\{\emptyset\}$ as the only partitioned subset of the empty set, but otherwise exclude the empty set from partitions.) We typically denote partitioned subsets by boldface and the underlying carrier sets in the usual italics. We define a *partial order* on the partitioned subsets by $\mathbf{Q} \blacktriangleleft \mathbf{P}$ if every partition member of $\mathbf{Q}$ is contained in some (necessarily unique) partition member of **P**. Partitioned subsets **P** and **Q** have a greatest lower bound $\mathbf{P} \wedge \mathbf{Q}$ given by all nonempty pairwise intersections of a member of **P** with a member of **Q**; the carrier set of $\mathbf{P} \wedge \mathbf{Q}$ is $P \cap Q$, the intersection of the carrier sets of **P** and **Q**, respectively.

If $\mathcal{P}$ is a paving, then a *$\mathcal{P}$-partitioned subset* is one for which each member of the partition belongs to $\mathcal{P}$. We denote by $\mathcal{P}^f$ all $\mathcal{P}$-partitioned subsets of Ω for which the partition is finite. We warn the reader that $\mathbf{P} \in \mathcal{P}^f$ does not necessitate that its carrier P is in $\mathcal{P}$, only that P is a finite disjoint union of members of $\mathcal{P}$.

A paving $\mathcal{K}$ is *monocompact* if every descending sequence

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$$

of nonempty members of $\mathcal{K}$ has a nonempty intersection. The following gives an alternative useful formulation of monocompactness.

Lemma 2.1. *Let $\mathcal{K}$ be a monocompact paving and let $\mathbf{K}_1 \blacktriangleright \mathbf{K}_2 \blacktriangleright \cdots$ be a decreasing sequence of partitioned subsets in $\mathcal{K}^f$. If each K_n is nonempty, then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.*

Proof. We define a graph as follows: a vertex is any member of any of the partitions $\mathbf{K}_n$ for $1 \leqslant n < \infty$ and an edge from a partition member A in $\mathbf{K}_n$ to B in $\mathbf{K}_{n+1}$ exists if $B \subseteq A$. The graph consists of finitely many trees, one for each partition member of $\mathbf{K}_1$, is infinite since each K_n is nonempty, and is finitely branching since each $\mathbf{K}_n \in \mathcal{K}^f$. Then one of the trees must be infinite. We apply König's lemma, which asserts that a finitely branching infinite tree must have an infinite branch. The intersection of vertices of this branch is nonempty by monocompactness and clearly contained in $\bigcap_{n=1}^{\infty} K_n$. $\square$

Let $\mathcal{A}$ be an algebra of subsets of Ω, and let $\lambda : \mathcal{A} \to [0, \infty]$ be a finitely additive measure. For a paving $\mathcal{P} \subseteq \mathcal{A}$, we say that $A \in \mathcal{A}$ *is approximated by* $\mathcal{P}^f$ if $\lambda(A) = \sup\{\lambda(P) \colon \mathbf{P} \in \mathcal{P}^f,\ P \subseteq A\}$. On the other hand, an arbitrary paving $\mathcal{K}$ is said to *approximate* $\mathcal{Q} \subseteq \mathcal{A}$ if for each $A \in \mathcal{Q}$ and $\varepsilon > 0$, there exists $K \in \mathcal{K}$ and $B \in \mathcal{Q}$ such that $B \subseteq K \subseteq A$ and $\lambda(A) - \lambda(B) < \varepsilon$.

Lemma 2.2. *Let $\lambda : \mathcal{A} \to [0, \infty]$ be a finitely additive measure on an algebra of subsets of Ω, and let $A_1 \supseteq A_2 \supseteq \cdots$ be a descending sequence of sets in $\mathcal{A}$ such that $\lambda(A_1) < \infty$. Suppose that $\mathcal{P} \subseteq \mathcal{A}$ is a paving closed under finite intersections such that each A_n is approximated by $\mathcal{P}^f$ and that $\mathcal{K}$ is a paving approximating $\mathcal{P}$. Then for $\varepsilon > 0$, sequences $(\mathbf{P}_n)$ and $(\mathbf{Q}_n)$ in $\mathcal{P}^f$ and $(\mathbf{K}_n)$ in $\mathcal{K}^f$ with carrier sets (P_n), (Q_n) and (K_n) respectively may be chosen so that the diagrams*

$$\begin{array}{ccccccccc}
A_1 & & \supseteq & & A_2 & & \supseteq & & A_3 \supseteq \cdots \\
\cup| & & & & \cup| & & & & \cup| \quad \cdots \\
Q_1 & & \supseteq & & Q_2 & & \supseteq & & Q_3 \supseteq \cdots \\
\cup| & & & & \cup| & & & & \cup| \quad \cdots \\
K_1 & \supseteq & P_1 & \supseteq & K_2 & \supseteq & P_2 & \supseteq & K_3 \supseteq \cdots
\end{array}$$

and

$$\mathbf{K}_1 \blacktriangleright \mathbf{P}_1 \blacktriangleright \mathbf{K}_2 \blacktriangleright \mathbf{P}_2 \blacktriangleright \mathbf{K}_3 \blacktriangleright$$

are valid, $\lambda(P_n) > \lambda(A_n) - \varepsilon$ *for each* n*, and hence*

$$\lim_{n\to\infty} \lambda(P_n) \geqslant \lim_{n\to\infty} \lambda(A_n) - \varepsilon.$$

Proof. Assume that $\mathbf{K}_j$, $\mathbf{P}_j$ and $\mathbf{Q}_j$ have been chosen for all $j < n$ such that $\lambda(A_j) - \lambda(P_j) < \varepsilon$ and such that the initial sections of the diagrams with subscripts below n are valid. Then

$$\begin{aligned}\lambda(A_n) - \lambda(A_n \cap P_{n-1}) &= \lambda(A_n \setminus P_{n-1}) \leqslant \lambda(A_{n-1} \setminus P_{n-1}) \\ &= \lambda(A_{n-1}) - \lambda(P_{n-1}) < \varepsilon.\end{aligned}$$

Let

$$\delta = \varepsilon - \lambda(A_n) + \lambda(A_n \cap P_{n-1}) > 0.$$

We can approximate A_n with some $\mathbf{Q}_n$ in $\mathcal{P}^f$ such that $Q_n \subseteq A_n$ and $\lambda(A_n) - \lambda(Q_n) < \delta$. Then applying the inclusion-exclusion principle to $A_n \cap P_{n-1}$ and Q_n, we have

$$\begin{aligned}&\lambda(A_n) - \lambda(P_{n-1} \cap Q_n) \\ &\quad = \lambda(A_n) - \lambda\big((A_n \cap P_{n-1}) \cap Q_n\big) \\ &\quad = \lambda(A_n) - \lambda(A_n \cap P_{n-1}) - \lambda(Q_n) + \lambda\big((A_n \cap P_{n-1}) \cup Q_n\big) \\ &\quad \leqslant \lambda(A_n) - \lambda(A_n \cap P_{n-1}) - \lambda(Q_n) + \lambda(A_n) \\ &\quad < \lambda(A_n) - \lambda(A_n \cap P_{n-1}) + \delta = \varepsilon.\end{aligned}$$

By hypothesis each member of $\mathbf{P}_{n-1} \wedge \mathbf{Q}_n$ belongs to $\mathcal{P}$. We apply the hypothesis that $\mathcal{K}$ approximates $\mathcal{P}$ and hence that

$$\lambda(P_{n-1} \cap Q_n) = \sup\big\{\lambda(P)\colon\ \mathbf{P} \blacktriangleleft \mathbf{K} \blacktriangleleft \mathbf{P}_{n-1} \wedge \mathbf{Q}_n,\ \mathbf{P} \in \mathcal{P}^f,\ \mathbf{K} \in \mathcal{K}^f\big\}$$

to find $\mathbf{P}_n \blacktriangleleft \mathbf{K}_n \blacktriangleleft \mathbf{P}_{n-1} \wedge \mathbf{Q}_n$ such that $\lambda(A_n) - \lambda(P_n) < \varepsilon$. Note that

$$\mathbf{P}_n \blacktriangleleft \mathbf{K}_n \blacktriangleleft \mathbf{P}_{n-1} \wedge \mathbf{Q}_n \blacktriangleleft \mathbf{P}_{n-1} \blacktriangleleft \mathbf{K}_{n-1}$$

implies that the appropriate inequalities continue to hold through step n in the second diagram of the lemma. Since additionally

$$P_n \subseteq K_n \subseteq P_{n-1} \cap Q_n \subseteq A_n,$$

the first diagram also remains valid. This completes our construction. The last assertion of the lemma then follows directly from the one preceding it. □

Corollary 2.3. *Under the assumptions of the preceding lemma and the additional assumptions that* $\mathcal{K}$ *is a monocompact paving and* $\bigcap_{n=1}^{\infty} A_n = \emptyset$*, then* $\lim_{n\to\infty} \lambda(A_n) = 0$.

Proof. Since $(\lambda(A_n))$ is a decreasing sequence of nonnegative numbers, the limit exists. Suppose that it is equal to $\eta > 0$. Pick $\varepsilon > 0$ such that $\varepsilon < \eta$. Pick the sequences $(\mathbf{K}_n)$ and $(\mathbf{P}_n)$ as in the preceding lemma. Then for each n, $\lambda(P_n) \geqslant \eta - \varepsilon > 0$, and thus each P_n, and in particular each K_n, is nonempty. But $\bigcap_n K_n \subseteq \bigcap_n A_n = \emptyset$, in contradiction to Lemma 2.1. □

The following is a mild, but for us useful, variant of a theorem of Topsøe (see [17, Theorem 6.1]).

Theorem 2.4. *Let $\mathcal{A}$ be an algebra of subsets of Ω and let $\lambda : \mathcal{A} \to [0,\infty]$ be a finitely additive measure. Suppose that $\mathcal{B} \subseteq \mathcal{A}$ is a semiring, that λ is finite on members of $\mathcal{B}$, that $\mathcal{P} \subseteq \mathcal{A}$ is a paving closed under finite intersections such that $\mathcal{P}^f$ approximates $\mathcal{B}$, and that $\mathcal{K}$ is a monocompact paving that approximates $\mathcal{P}$:*

$$\begin{array}{llcl} \mathcal{K}\ (\textit{monocompact})\ \textit{approximates}\ \ \mathcal{P} & & \subseteq & \mathcal{A}\ (\textit{an algebra}) \\ & & & \cup| \\ & \mathcal{P}^f\ \ \textit{approximates} & & \mathcal{B}\ (\textit{a semiring}) \end{array}$$

Then λ restricted to $\mathcal{B}$ extends to a σ-additive measure on a σ-algebra containing $\mathcal{B}$. This measure is unique on the σ-ring generated by $\mathcal{B}$.

Proof. Since λ is finite on $\mathcal{B}$, it follows from the preceding corollary that for any descending sequence (A_n) in $\mathcal{B}$ with empty intersection, we have $\lim_n \lambda(A_n) = 0$. It is then standard in measure theory that this implies λ is σ-additive on $\mathcal{B}$, and hence extends to a σ-additive measure on a σ-algebra containing $\mathcal{B}$ (see the preceding section, particularly Corollary 1.2). $\square$

Remark 2.5. In various circumstances the preceding machinery can yield inner regularity results about the extension of λ to a σ-additive measure. For example, if additionally $\mathcal{A} = \mathcal{B}$ in the hypotheses of Theorem 2.4, then using Lemma 2.2, the constant sequence $A = A_1 = A_2 = \cdots$, and the countable additivity of the extension of λ, one sees that any member of $\mathcal{A}$ can be approximated arbitrarily closely from the inside by a set that is a countable decreasing intersection of sets that are finite disjoint unions of members of $\mathcal{K}$. Furthermore, this intersection is in the σ-algebra generated by $\mathcal{A}$, since it follows from Lemma 2.2 that $\bigcap_{n=1}^{\infty} K_n = \bigcap_{n=1}^{\infty} P_n$.

3. Crescent outer measures

Our interest in this paper lies particularly in the topological setting. Let X be a topological space and $\mathcal{O}(X)$ the lattice of open sets. There is a naturally associated semialgebra $\mathcal{SO}(X)$ consisting of all locally closed sets, sets of the form $U \setminus V$, for U, V open. We call these sets *crescents*. Their finite disjoint unions constitute the smallest algebra containing $\mathcal{O}(X)$, called the *crescent algebra*, and denoted $\mathcal{AO}(X)$.

Let $\lambda : \mathcal{O}(X) \to [0,\infty]$ be a valuation on the open set lattice of X. We fix some finitely additive extension of λ to the crescent algebra, guaranteed by the Smiley–Horn–Tarski theorem, and continue to denote it by λ.

We define the *crescent outer measure* on all subsets A of X by

$$\lambda^*(A) = \inf\left\{ \sum_{i=1}^{\infty} \lambda(S_i) \colon\ A \subseteq \bigcup_{i=1}^{\infty} S_i \text{ and } S_i \in \mathcal{AO}(X) \text{ for all } i \right\}.$$

Recall that an *outer measure* on a set X is a function $\lambda^*: 2^X \to [0,\infty]$ that is strict, monotone, and countably subadditive (*countable subadditivity* means that $\lambda^*(A) \leqslant \sum_{i=1}^{\infty} \lambda^*(A_i)$, whenever A is contained in the union of the A_i). Typically the outer measure for topological spaces is defined in terms of open covers, but for non-Hausdorff spaces the crescent algebra is much better suited. Note that since each member of the crescent algebra is a finite disjoint union of crescents, one obtains the same outer measure if one defines the outer measure by replacing coverings by members of the crescent algebra with coverings by crescents. It is standard that the constructions of the preceding type result in outer measures.

Lemma 3.1. *The crescent outer measure λ^* is indeed an outer measure.*

One uses the crescent outer measure and the Carathéodory criterion to define measurability: A subset A of X is λ^*-*measurable* if, for all $E \subseteq X$,

$$\lambda^*(E) = \lambda^*(A \cap E) + \lambda^*\big(A^c \cap E\big).$$

The following theorem is standard; indeed it occurs as part of the proof of the earlier cited Theorem 1.1.

Theorem 3.2. *If λ^* is an outer measure on all subsets of a set Ω, then the measurable sets form a σ-algebra and λ^* restricted to this σ-algebra is a measure. If λ^* arises as the outer measure coming from a countably additive measure λ on a semialgebra of sets, then the members of the semialgebra are all measurable and the outer measure λ^* extends λ.*

In particular, in the topological setting where $\mathcal{A}$ is the crescent algebra of a topological space X and λ is countably additive on the semialgebra of crescents, then it extends to a Borel measure on the Borel algebra, namely λ^* restricted to the Borel algebra. If λ is also finite, then this extension is unique.

A valuation λ on the lattice of open sets of a topological space X is *locally finite* if every point of X has an open neighbourhood U with $\lambda(U) < \infty$. It is standard practice in measure theory that many results involving finite measures extend to σ-finite or locally finite measures. We present some results along this line suitable for our context. Our development resembles that of Alvarez-Manilla in [2, Section 2.3].

Lemma 3.3. *Let $\lambda: \mathcal{L} \to [0,\infty]$ be a valuation on a lattice of subsets of X containing the empty set. Suppose $(U_j)_{j\in D}$ is a directed family in $\mathcal{L}$ with $X = \bigcup_j U_j$, $\lambda(U_j) < \infty$ for each U_j. If for each j, the valuation $\lambda_j(U) := \lambda(U_j \cap U)$ on $\mathcal{L}$ extends to a* (*necessarily finite*) *measure on the σ-algebra $\mathcal{A}_\sigma(\mathcal{L})$ generated by $\mathcal{L}$, then λ gives rise to a measure $\overline{\lambda}$ on $\mathcal{A}_\sigma(\mathcal{L})$ that extends λ on each $A \in \mathcal{L}$ such that $\lambda(A) = \lim_j \lambda_j(A)$.*

Proof. By the Smiley–Horn–Tarski theorem we extend λ to a finitely additive measure (still called λ) on the algebra of sets $\mathcal{A}$ generated by $\mathcal{L}$. For each j, we have that $\lambda_j(A) := \lambda(A \cap U_j)$ is a finitely additive measure on $\mathcal{A}$, an extension of λ_j from $\mathcal{L}$ to $\mathcal{A}$. (Since λ_j is finite, this extension is unique.) Let λ_j^* be the outer measure on 2^X arising from the extension of λ_j to $\mathcal{A}$. Since by hypothesis λ_j extends to a measure on $\mathcal{A}_\sigma(\mathcal{L})$, we conclude that λ_j is countably additive on $\mathcal{A}$, and hence by the preceding theorem each member of $\mathcal{A}_\sigma(\mathcal{L})$ is λ_j^*-measurable and the restriction of λ_j^* to $\mathcal{A}_\sigma(\mathcal{L})$ is a measure extending λ_j.

For $U_j \subseteq U_k$, we have for any $A \in \mathcal{A}$,

$$\lambda_j(A) = \lambda(A \cap U_j) \leqslant \lambda(A \cap U_k) = \lambda_k(A).$$

It follows that $\lambda_j^* \leqslant \lambda_k^*$, and hence that $(\lambda_j^*(A))$ is a directed net for each $A \subseteq X$. Set $\bar{\lambda}(A) := \lim_j \lambda_j^*(A) = \sup\{\lambda_j^*(A)\colon\ j \in D\}$. It is straightforward to verify that $\bar{\lambda}$ is an outer measure. For any $A \in \mathcal{A}_\sigma(\mathcal{L})$ and arbitrary set E, we have

$$\begin{aligned}\bar{\lambda}(E) &= \lim_j \lambda_j^*(E) = \lim_j \lambda_j^*(A \cap E) + \lim_j \lambda_j^*\big(A \cap E^c\big)\\ &= \bar{\lambda}(A \cap E) + \bar{\lambda}\big(A \cap E^c\big),\end{aligned}$$

since A is λ_j^*-measurable for each j. Hence A is also $\bar{\lambda}$-measurable. Thus $\mathcal{A}_\sigma(\mathcal{L})$ is a sub-σ-algebra of the $\bar{\lambda}$-measurable sets, and hence $\bar{\lambda}$ restricted to $\mathcal{A}_\sigma(\mathcal{L})$ is a measure.

If $U \in \mathcal{L}$ satisfies $\lambda(U) = \lim_j \lambda_j(U)$, then

$$\bar{\lambda}(U) = \lim_j \lambda_j^*(U) = \lim_j \lambda_j(U) = \lambda(U). \qquad \square$$

Remark 3.4. The collection $\mathcal{A}_1$ of sets A with the property that $U_j \cap A \in \mathcal{A}_\sigma(\mathcal{L})$ for each j is easily seen to be a σ-algebra containing $\mathcal{A}_\sigma(\mathcal{L})$. Furthermore, each of these sets is $\bar{\lambda}$-measurable, since for an arbitrary subset E the argument of the preceding proof can be extended as follows:

$$\begin{aligned}\bar{\lambda}(E) &= \lim_j \lambda_j^*(E) = \lim_j \lambda_j^*\big((U_j \cap A) \cap E\big) + \lim_j \lambda_j^*\big((U_j \cap A) \cap E^c\big)\\ &= \lim_j \lambda_j^*(A \cap E) + \lim_j \lambda_j^*\big(A \cap E^c\big) = \bar{\lambda}(A \cap E) + \bar{\lambda}\big(A \cap E^c\big).\end{aligned}$$

Therefore $\bar{\lambda}$ restricted to $\mathcal{A}_1$ is a measure.

In our topological context we will need a continuity condition for the valuations we consider: A valuation $\lambda : \mathcal{O}(X) \to [0, +\infty]$ is called *continuous* if $\lambda(\bigcup_j U_j) = \sup_j \lambda(U_j)$ for all directed families (U_j) in $\mathcal{O}(X)$. The next corollary is a special case of the preceding lemma.

Corollary 3.5. *Let $\lambda : \mathcal{O}(X) \to [0, \infty]$ be a continuous valuation on the open set lattice of a topological space X. Suppose $(U_j)_{j \in D}$ is a directed family of open sets with $X = \bigcup_j U_j$, $\lambda(U_j) < \infty$ for each U_j, and the restriction valuation $\lambda_j(U) := \lambda(U_j \cap U)$ extends to a* (*finite*) *Borel measure for each j. Then λ extends to a Borel measure on X.*

Remark 3.6. Note in the previous corollary that if λ is a continuous valuation, then its restriction to any open set will again be a continuous valuation. We also note that there are equivalent ways of extending λ_j to a Borel measure: one can either do it on all of X or one can first extend it to a Borel measure on the subspace U_j, and then define the extension on all Borel sets by taking the measure of the intersection of the Borel set with U_j.

Corollary 3.7. *If X is a topological space and $\lambda : \mathcal{O}(X) \to [0, \infty]$ is a locally finite continuous valuation, and if the finite continuous valuations $\lambda_V(U) := \lambda(V \cap U)$ for $\lambda(V) < \infty$ on $\mathcal{O}(X)$ extend to Borel measures on the Borel σ-algebra, then $\lambda : \mathcal{O}(X) \to [0, \infty]$ also extends to a Borel measure on the Borel σ-algebra.*

Proof. All finite unions of λ-finite open sets form a directed collection satisfying the hypotheses of Corollary 3.5. □

4. Directed limits of simple measures

In this section we derive a quite novel extension result due to Alvarez-Manilla et al. [4] from our generalization of Topsøe's theorem. A major motivation for this theorem comes out of domain theory, where one has the important result that any continuous valuation on a domain can be represented as a directed supremum of simple valuations (see, for example, [8, Section IV-9]). Thus all continuous valuations on the Scott topology of a domain admit extensions to Borel measures. Moreover, the result may then be extended to subspaces of domains that are measurable subsets by defining a new valuation on the open subsets of the domain to be the value of the given valuation on the intersection of the open subset with the subspace. This applies in particular to those situations where we can represent a topological space as the G_δ-subspace of maximal points of a domain (see [8, Section V-6]). One extends the result to all complete metric spaces in this fashion for example.

We always assume our spaces to be T_0-spaces, which means that different points have distinct neighborhood systems. A T_0-space X always carries a natural partial order, the *order of specialization* $\leqslant$ which may be defined in various equivalent ways: $x \leqslant y$ if $x \in \overline{\{y\}}$, the closure of the singleton set $\{y\}$, or, equivalently, if every open set U containing x also contains y. Every open set is an upper set with respect to this order and every upper set is an intersection of open sets. Upper sets are also called *saturated*. Similarly, every closed set is a lower set and every lower set is a union of closed sets. It follows that every crescent $C = U \setminus V$ is order convex, that is, $a \leqslant b \leqslant c$ and $a, c \in C$ imply $b \in C$.

Let X be a *monotone convergence space*, a T_0-space for which every subset directed in the order of specialization has a supremum to which it converges. On a monotone convergence space we may introduce the *Scott topology*, which is finer than the given topology, by saying that U is Scott-open if U is an upper set and if, for any directed subset D, one has $D \cap U \neq \emptyset$ whenever $\sup D \in U$.

We order the valuations on $\mathcal{O}(X)$, the lattice of open sets, by $\lambda \leqslant \mu$ if $\lambda(U) \leqslant \mu(U)$ for all open sets U. This order is sometimes called the *stochastic order*.

For $x \in X$, a topological space, the *point valuation* $\eta_x : \mathcal{O}(X) \to [0, +\infty[$ is defined by $\eta_x(U) = 1$ if $x \in U$ and $\eta_x(U) = 0$ if $x \notin U$. A valuation on X is *simple* if it is a finite linear combination $\sum_{i=1}^{n} r_i \eta_{x_i}$, $r_i \in [0, +\infty[$, of point valuations. A net $\{\lambda_\alpha\colon \alpha \in D\}$ of simple valuations is *directed* if $\alpha \leqslant \beta$ implies $\lambda_\alpha \leqslant \lambda_\beta$. We say that a valuation λ is the directed supremum of simple valuations if there exists a directed net (λ_α) of simple valuations such that $\lambda(U) = \sup\{\lambda_\alpha(U)\colon \alpha \in D\}$ for all open sets U.

We assume in the rest of this section, unless explicitly stated otherwise, that (λ_α) is a directed net of simple valuations with supremum the valuation λ, a valuation taking only finite values.

Lemma 4.1. *If $\lambda_\alpha \leqslant \lambda_\beta$ are simple valuations and A is an upper set with respect to the specialization order, then $\lambda_\alpha(A) \leqslant \lambda_\beta(A)$.*

Proof. Since A is the directed intersection of the open sets containing it and since the supports of λ_α and λ_β are both finite, for all sufficiently small U containing A, we have

$$\lambda_\alpha(A) = \lambda_\alpha(U) \leqslant \lambda_\beta(U) = \lambda_\beta(A). \qquad \square$$

Definition 4.2. Let (λ_α) be a directed net of simple valuations. Then each λ_α defines a measure on the whole power set of X. Define $\bar{\lambda}(A) = \lim_\alpha \lambda_\alpha(A)$, provided the limit exists.

Proposition 4.3. *The function $\bar{\lambda}$ extends λ, is defined on the algebra generated by the upper sets, and is finitely additive on this algebra.*

Proof. It is immediate from the fact that λ is the supremum of the λ_α on all open sets U that $\bar{\lambda}$ extends λ. For any upper set A, it follows from Lemma 4.1 that $(\lambda_\alpha(A))$ is directed, hence convergent in $[0,\infty)$, since it is bounded above by $\lambda(X)$. The upper sets form a lattice containing X. Hence any A in the generated algebra can be written as a finite disjoint union $A = \bigcup_{i=1}^n A_i \setminus B_i$, where A_i and B_i are upper sets and $B_i \subseteq A_i$. Then

$$\begin{aligned}\sum_{i=1}^n \bar{\lambda}(A_i) - \bar{\lambda}(B_i) &= \sum_{i=1}^n \lim_\alpha \bigl(\lambda_\alpha(A_i) - \lambda_\alpha(B_i)\bigr) = \lim_\alpha \sum_{i=1}^n \lambda_\alpha(A_i \setminus B_i) \\ &= \lim_\alpha \lambda_\alpha\left(\bigcup_{i=1}^n A_i \setminus B_i\right) = \lim_\alpha \lambda_\alpha(A).\end{aligned}$$

Thus $\bar{\lambda}(A)$ is defined. Essentially the same computation establishes that $\bar{\lambda}$ is finitely additive. $\square$

Theorem 4.4. *Let X be a monotone convergence space and let λ be a finite valuation on $\mathcal{O}(X)$ that is the supremum of a directed net of simple valuations (λ_α). Then λ extends to a σ-additive measure on a σ-algebra containing the Scott-open sets, and hence the original open sets.*

Proof. Let $\bar{\lambda}$ be the extension of λ to the algebra $\mathcal{A}$ generated by the upper sets, as in the previous proposition. Let $\mathcal{B}$ be the subalgebra generated by the Scott-open sets; note that the Scott-open sets contain all the open sets since X is a monotone convergence space. Let $\mathcal{P}$ be the paving consisting of all $A \setminus U$, where A is an upper set and U is a Scott-open set. Since $(A_1 \setminus U_1) \cap (A_2 \setminus U_2) = (A_1 \cap A_2) \setminus (U_1 \cup U_2)$, the paving $\mathcal{P}$ is closed under finite intersections. Furthermore, any member of $\mathcal{B}$ is a finite disjoint union of crescents, sets of the form $U \setminus V$, where U and V are Scott-open; thus the set of carriers of $\mathcal{P}^f$ contains all members of $\mathcal{B}$, hence approximates it.

Let $\mathcal{K}$ denote the paving of all sets of the form $\uparrow F \setminus U$, where U is Scott-open, F is a finite set and where $\uparrow F = \{y \in X\colon\ x \leqslant y \text{ for some } x \in F\}$ denotes the upper set generated by F with respect to the specialization order. It follows from an application of either Rudin's lemma (see [8, Lemma III-3.3]) or König's lemma that $\mathcal{K}$ is monocompact.

Consider any $A \setminus U$, where A is an upper set and U is Scott-open. For any upper set A, by Lemma 4.1 the $\lambda_\alpha(A)$ converge upwards to $\bar{\lambda}(A)$. Thus for $\varepsilon > 0$, there exists α such that $0 \leqslant \bar{\lambda}(A) - \lambda_\alpha(A) < \varepsilon$. Since λ_α is simple, the intersection of its support with A is a finite set F, and $\lambda_\alpha(A) = \lambda_\alpha(\uparrow F)$. Then

$$\bar{\lambda}(A) - \varepsilon < \lambda_\alpha(A) = \lambda_\alpha(\uparrow F) \leqslant \bar{\lambda}(\uparrow F).$$

Additionally we have $\bar{\lambda}(\uparrow F \cap U) \leqslant \bar{\lambda}(A \cap U)$ and thus

$$\bar{\lambda}(\uparrow F \setminus U) = \bar{\lambda}(\uparrow F) - \bar{\lambda}(\uparrow F \cap U) > \bar{\lambda}(A) - \varepsilon - \bar{\lambda}(A \cap U) = \bar{\lambda}(A \setminus U) - \varepsilon.$$

We conclude that $\mathcal{K}$ approximates $\mathcal{P}$. Note we do not need to choose a further element in $\mathcal{P}$, since $\mathcal{K} \subseteq \mathcal{P}$ already. It follows now by Theorem 2.4 that the extension of λ to $\mathcal{B}$, which is just $\bar{\lambda}$ restricted to $\mathcal{B}$, has a σ-additive extension to a σ-algebra containing $\mathcal{B}$. □

Corollary 4.5. *Let X be a monotone convergence space and let λ be a locally finite valuation on $\mathcal{O}(X)$ that is the supremum of a directed net of simple valuations (λ_α). Then λ extends to a Borel measure on the Borel sets.*

Proof. We first note that a directed supremum of simple valuations must be continuous (use the simple valuations to approximate any open set with a finite set from inside; then any open set containing the finite set will approximate the original open set). We can apply then the preceding theorem and Corollary 3.7, since the restrictions of λ to open sets will again be a directed supremum of the corresponding restrictions of the simple valuations. □

Note that in Theorem 4.4 the extension of the valuation λ to a countably additive measure is uniquely determined on the σ-algebra of Borel sets, and in Corollary 4.5 the extension is uniquely determined on the σ-ring generated by the open sets of finite value.

5. Locally compact spaces

In this section we shall see how to use Theorem 2.4 for extending a continuous valuation on the open set lattice of a locally compact sober space to a Borel measure. It is essential that we do not assume our spaces to be Hausdorff. The main theorem in the generality given here was first proved by Alvarez-Manilla [2,3] in an alternative fashion using earlier theorems of Topsøe. The case of locally compact sober spaces with a countable base of open sets was carried out earlier in [11].

Soberness is a kind of completeness property for topological spaces: A space X is *sober*, if every irreducible closed subset is the closure of a unique point, where a closed set is called irreducible if it is nonempty and not the union of two of its proper closed subsets. Every sober space is T_0. In a sober space every subset that is directed with respect to the specialization order has a least upper bound and it converges to this point. Some basic information on sober spaces can be found in [8, Section 0-5].

A subset of a topological space is called a *lens* if it is of the form $K = Q \setminus U$, where Q is a compact saturated subset and U an open subset of X. We denote by $\mathcal{K}$ the set of all lenses. From [8, Lemma VI-6.1], we quote the following:

Lemma 5.1. *In a sober space, the intersection of every filtered* ($=$ *down-directed*) *family of nonempty lenses is nonempty. In particular, the lenses form a monocompact paving* $\mathcal{K}$.

We now consider a locally compact space, that is, a space in which every point has neighborhood basis of compact neighborhoods, and a continuous valuation λ on the lattice $\mathcal{O}(X)$ with finite values. As in Section 3 we extend λ to a finitely additive measure on the crescents and on the crescent algebra.

Lemma 5.2. *The paving $\mathcal{K}$ of lenses approximates λ on the semialgebra $\mathcal{S} = \mathcal{SO}(X)$ of crescents $U \setminus V$, where $U, V \in \mathcal{O}(X)$.*

Proof. The argument is quite standard: Because of local compactness a given open set U is the union of a directed family of open sets U_j with the property that there is a compact saturated set Q_j such that $U_j \subseteq Q_j \subseteq U$. As the valuation λ is supposed to be continuous, $\lambda(U) = \sup_j \lambda(U_j)$ whence, for any given $\varepsilon > 0$, there is a j such that $\lambda(U_j) \geqslant \lambda(U) - \varepsilon$. If now we consider a crescent $U \setminus V$ where V is an open set contained in U, then with the above choice of j we have $\lambda(U_j \setminus V) > \lambda(U \setminus V) - \varepsilon$ and $U_j \setminus V \subseteq Q_j \setminus V \subseteq U \setminus V$. $\square$

We are now ready for the main theorem of this section:

Theorem 5.3. *Every locally finite continuous valuation on a locally compact sober space can be extended to a Borel measure on the Borel σ-algebra.*

Proof. We first suppose that λ is a finite continuous valuation. We extend it to the crescent algebra $\mathcal{A}$ by the Smiley–Horn–Tarski theorem. For $\mathcal{K}$ the paving of lenses, $\mathcal{P}$ the semialgebra of crescents, and $\mathcal{A} = \mathcal{B}$, the hypotheses of Theorem 2.4 are satisfied by Lemmas 5.2 and 5.1; thus we may extend λ to a Borel measure. By Corollary 3.7 this result extends to locally finite continuous valuations. $\square$

We remark that sobriety is necessary in the preceding theorem. If one considers the set $\mathbb{N}$ equipped with the topology for which the nonempty open sets are of the form $[n, \infty)$, $n \in \mathbb{N}$, then the valuation which assigns to each nonempty open set the value 1 is a finite continuous valuation that does not extend to a Borel measure.

6. Coherent spaces

In Hausdorff spaces, the intersection of two compact sets is always compact. This is no longer true in T_0-spaces even if these compact sets are saturated. In this section we restrict ourselves to T_0-spaces with the property that the intersection of two compact saturated subsets is always compact; these spaces will be called *coherent*.

A Hausdorff space is a coherent sober space, and there is no distinction between the compact sets, the compact saturated sets, the lenses, and the family they generate under finite unions and arbitrary intersections. However, for the theory of T_0-spaces all are typically distinct.

In the remainder of this section we suppose X to be a coherent sober space. By coherence, the collection $\mathcal{Q}(X)$ of compact saturated subsets is a lattice under finite unions and finite intersections and, by soberness, also stable for intersections of arbitrary nonempty families (*see* [8, II-1.22]). *We adopt the convention that the letters $U, U_1, U_2, \ldots$ always denote open sets, i.e., arbitrary members of $\mathcal{O}(X)$ and the letters Q, Q_1, $Q_2, \ldots$ compact saturated sets, i.e., arbitrary members of $\mathcal{Q}(X)$.*

We will exploit the duality COMPACT-OPEN. Accordingly, this section will have two parts: Firstly we start with a finite valuation

$$\lambda : \mathcal{Q}(X) \to [0, +\infty[,$$

and secondly we begin with a locally finite valuation

$$\mu : \mathcal{O}(X) \to [0, +\infty].$$

In both cases we want to extend the valuations to measures on the σ-algebra generated by the open sets together with the compact saturated sets. In the earlier sections our aim was more modest: we were looking for an extension of a valuation to a measure on the Borel sets, that is, on the σ-algebra generated by the open sets alone. The results for the first case, in particular Lemmas 6.2 and 6.3 are due to Norberg and Vervaat [14], and we use some of their ideas. But note also the remarks at the end of Section 7.

Our motivation for considering this larger σ-algebra comes from the following: In a coherent sober space, the compact saturated sets together with the whole space X are the closed sets of a topology called the *co-compact topology*. We want to obtain a measure defined for the Borel sets of both the original and the co-compact topology. The coarsest common refinement of the co-compact topology and the original topology is called the *patch topology*. It need not be included in the σ-algebra just mentioned. In the next section we will extend the results of this section to the Borel sets of the patch topology. The co-compact as well as the patch topology play an important role in domain theory (see [8]).

Let us begin with a valuation $\lambda : \mathcal{Q}(X) \to [0, +\infty[$. We first extend the set function λ to all subsets A of X by defining

$$\lambda_*(A) := \sup\bigl\{\lambda(Q)\colon\ Q \in \mathcal{Q}(X) \text{ and } Q \subseteq A\bigr\}.$$

Lemma 6.1. *The extension λ_* is supermodular, that is, for all subsets A_1 and A_2, one has*

$$\lambda_*(A_1) + \lambda_*(A_2) \leqslant \lambda_*(A_1 \cup A_2) + \lambda_*(A_1 \cap A_2).$$

Proof. The proof results from the following easy calculations: For all $A_1, A_2 \subseteq X$, we have:

$$\lambda_*(A_1 \cap A_2) = \sup_{Q \in A_1 \cap A_2} \lambda(Q) = \sup_{Q_1 \subseteq A_1, Q_2 \subseteq A_2} \lambda(Q_1 \cap Q_2) \tag{1}$$

and

$$\lambda_*(A_1 \cup A_2) = \sup_{Q \subseteq A_1 \cup A_2} \lambda(Q) \geqslant \sup_{Q_1 \subseteq A_1,\, Q_2 \subseteq A_2} \lambda(Q_1 \cup Q_2), \tag{2}$$

and consequently

$$\begin{aligned}
\lambda_*(A_1) + \lambda_*(A_2) &= \sup_{Q_1 \subseteq A_1} \lambda(Q_1) + \sup_{Q_2 \subseteq A_2} \lambda(Q_2) && (3)\\
&= \sup_{Q_1 \subseteq A_1,\, Q_2 \subseteq A_2} \lambda(Q_1) + \lambda(Q_2) && (4)\\
&= \sup_{Q_1 \subseteq A_1,\, Q_2 \subseteq A_2} \lambda(Q_1 \cup Q_2) + \lambda(Q_1 \cap Q_2) && (5)\\
&= \sup_{Q_1 \subseteq A_1,\, Q_2 \subseteq A_2} \lambda(Q_1 \cup Q_2) + \sup_{Q_1 \subseteq A_1,\, Q_2 \subseteq A_2} \lambda(Q_1 \cap Q_2) && (6)\\
&\leqslant \lambda_*(A_1 \cup A_2) + \lambda_*(A_1 \cap A_2) \quad \text{by (1), (2).} && (7)
\end{aligned}$$

This shows supermodularity. □

We notice that λ_* is continuous when restricted to the lattice $\mathcal{O}(X)$ of open sets in the sense that

$$\lambda_*(U) = \sup_j \lambda_*(U_j),$$

whenever U is the union of a directed family of open sets U_j. (Indeed, every compact saturated set contained in U is contained in some U_j.)

We now require *Wilker's condition* (see [19]) to hold: Whenever U_1 and U_2 are open sets and whenever Q is a compact saturated set contained in $U_1 \cup U_2$, then there are compact saturated sets $Q_1 \subseteq U_1$ and $Q_2 \subseteq U_2$ such that $Q \subseteq Q_1 \cup Q_2$.

Wilker's condition implies that, for open sets A_1 and A_2, the inequality can be replaced by an equality in (2) and, consequently, in (7). Thus, we have shown:

Lemma 6.2. *If X satisfies Wilker's condition, then λ_* is a continuous valuation on the lattice $\mathcal{O}(X)$ of open sets.*

In a dual way, we define another set function λ_{**} for all subsets A of X by

$$\lambda_{**}(A) := \inf\bigl\{\lambda_*(U)\colon\ A \subseteq U \in \mathcal{O}(X)\bigr\}.$$

Clearly,

$$\lambda_*(A) \leqslant \lambda_{**}(A) \quad \text{for all } A \subseteq X. \tag{8}$$

We consider the collection of sets for which λ_* and λ_{**} agree:

$$\begin{aligned}
&\mathcal{L} := \bigl\{A \subseteq X\colon\ \lambda_*(A) = \lambda_{**}(A)\bigr\} \quad \text{and}\\
&\mathcal{L}^f := \bigl\{A \subseteq X\colon\ \lambda_*(A) = \lambda_{**}(A) < \infty\bigr\}.
\end{aligned}$$

More explicitly, a set A belongs to $\mathcal{L}$ iff

$$\sup\bigl\{\lambda(Q)\colon\ A \supseteq Q \in \mathcal{Q}(X)\bigr\} = \inf\bigl\{\lambda_*(U)\colon\ A \subseteq U \in \mathcal{O}(X)\bigr\}.$$

By the definition of λ_*, all open sets belong to $\mathcal{L}$. The compact saturated sets also belong to $\mathcal{L}$, and hence to $\mathcal{L}^f$, if we require the following to hold:

$$\lambda(Q) = \inf\bigl\{\lambda_*(U)\colon\ U \in \mathcal{O}(X) \text{ and } Q \subseteq U\bigr\} \quad \text{for every } Q \in \mathcal{Q}(X). \tag{9}$$

This is tantamount to saying that, for every saturated compact set Q and every $\varepsilon > 0$, there is an open neighborhood U of Q with the property that $\lambda(Q') < \lambda(Q) + \varepsilon$ for every compact saturated set Q' inside U. We shall say that λ is *outer regular* on $\mathcal{Q}(X)$ if Eq. (9) holds.

Lemma 6.3. *If X satisfies Wilker's condition and λ is outer regular, then λ_* and λ_{**} agree on the lattice $\mathcal{L}^f$ of subsets of X, which contains $\mathcal{Q}(X)$ and all λ_*-finite members of $\mathcal{O}(X)$. Furthermore, λ_* is a valuation on $\mathcal{L}^f$.*

Proof. By Lemma 6.1, λ_* is supermodular, i.e., $\lambda_*(A_1) + \lambda_*(A_2) \leqslant \lambda_*(A_1 \cup A_2) + \lambda_*(A_1 \cap A_2)$ for all subsets A_1, A_2 of X. In a completely similar way, one sees that λ_{**} is submodular, i.e., $\lambda_{**}(A_1) + \lambda_{**}(A_2) \leqslant \lambda_{**}(A_1 \cup A_2) + \lambda_{**}(A_1 \cap A_2)$. Since $\lambda_* = \lambda_{**}$ on $\mathcal{L}^f$, the modularity follows, once we show that $\mathcal{L}^f$ is a lattice.

Let $A_1, A_2 \in \mathcal{L}^f$. We then have

$$\begin{aligned}
\lambda_{**}(A_1) + \lambda_{**}(A_2) &= \lambda_*(A_1) + \lambda_*(A_2) \quad \text{as } A_1, A_2 \in \mathcal{L} && (10)\\
&\leqslant \lambda_*(A_1 \cup A_2) + \lambda_*(A_1 \cap A_2) \quad \text{as } \lambda_* \text{ is supermodular} && (11)\\
&\leqslant \lambda_{**}(A_1 \cup A_2) + \lambda_{**}(A_1 \cap A_2) \quad \text{by (8)} && (12)\\
&\leqslant \lambda_{**}(A_1) + \lambda_{**}(A_2) \quad \text{as } \lambda_{**} \text{ is submodular.} && (13)
\end{aligned}$$

Hence, equality holds throughout; in particular,

$$\lambda_{**}(A_1 \cup A_2) + \lambda_{**}(A_1 \cap A_2) = \lambda_*(A_1 \cup A_2) + \lambda_*(A_1 \cap A_2) < +\infty.$$

As $\lambda_*(A_1 \cup A_2) \leqslant \lambda_{**}(A_1 \cup A_2)$ and $\lambda_*(A_1 \cap A_2) \leqslant \lambda_{**}(A_1 \cap A_2)$, we conclude that $\lambda_{**}(A_1 \cup A_2) = \lambda_*(A_1 \cup A_2)$ and $\lambda_{**}(A_1 \cap A_2) = \lambda_*(A_1 \cap A_2)$, whence $A_1 \cup A_2 \in \mathcal{L}^f$ and $A_1 \cap A_2 \in \mathcal{L}^f$. $\square$

In passing let us note two consequences of outer regularity. This condition firstly implies that λ is continuous on $\mathcal{Q}(X)$ in the following sense:

$$\lambda(Q) = \inf_j \lambda(Q_j),$$

whenever Q is the intersection of a filter basis of compact saturated sets Q_j. (Indeed, by the Hofmann–Mislove theorem [8, Theorem II-1.21], every open set containing Q contains some Q_j.) As λ was supposed to have finite values for compact saturated sets, it secondly implies that λ_* is locally finite, that is, every point (and consequently every compact set) has an open neighborhood U with $\lambda_*(U) < \infty$.

We note that $\mathcal{Q}(X)$ *approximates* λ_* *on* $\mathcal{L}^f$ *from inside* and that $\mathcal{O}(X)$ *approximates* λ_* *on* $\mathcal{L}^f$ *from outside* in the sense that for any $L, M \in \mathcal{L}^f$ and every $\varepsilon > 0$ there are a compact saturated set $Q \subseteq L$ and an open set $U \supseteq M$ such that

$$\lambda_*(L) - \lambda(Q) < \varepsilon \quad \text{and} \quad \lambda_*(U) - \lambda_*(M) < \varepsilon.$$

We now consider the semiring $\mathcal{S}$ arising from the lattice $\mathcal{L}^f$ that consists of all sets of the form $L \setminus M$ with $L, M \in \mathcal{L}^f$ and $M \subseteq L$. We may extend λ_* to $\mathcal{S}$ by defining $\lambda_*(L \setminus M) = \lambda_*(L) - \lambda_*(M)$.

Lemma 6.4. *The collection $\mathcal{K}$ of lenses $Q \setminus U$ is contained in the semiring $\mathcal{S}$. Furthermore, since $\mathcal{Q}(X)$ approximates λ_* on $\mathcal{L}$ from inside and $\mathcal{O}(X)$ approximates λ_* on $\mathcal{L}$ from outside, the collection $\mathcal{K}$ approximates λ_* from inside on the semiring $\mathcal{S}$.*

Proof. Consider any lens $Q \setminus U$. By outer regularity, there exists an open set V containing Q with $\lambda_* V < \infty$. Then $Q \setminus U = Q \setminus (U \cap V)$, and the latter is in $\mathcal{S}$.

If for given $M \subseteq L$ and $\varepsilon > 0$, we choose the ε-approximations U to M and Q to L as in the definitions, then $Q \setminus U \subseteq L \setminus M$ and $\lambda_*(L \setminus M) - \lambda_*(Q \setminus U) < 2\varepsilon$. The proof for the latter inequality is quite straightforward:

$$
\begin{aligned}
&\lambda_*(L \setminus M) - \lambda_*(Q \setminus U) \\
&\quad = \big(\lambda_*(L) - \lambda_*(M)\big) - \big(\lambda(Q) - \lambda_*(U \cap Q)\big) \\
&\quad = \big(\lambda_*(L) - \lambda(Q)\big) + \big(\lambda_*(U \cap Q) - \lambda_*(M)\big) \\
&\quad \leqslant \big(\lambda_*(L) - \lambda(Q)\big) + \big(\lambda_*(U) - \lambda_*(M)\big) < 2\varepsilon \quad \text{as } U \cap Q \subseteq U. \qquad \square
\end{aligned}
$$

Theorem 6.5. *Let X be a coherent sober space satisfying the Wilker condition. Then every outer regular valuation λ on the collection $\mathcal{Q}(X)$ of compact saturated sets can be extended to a locally finite countably additive measure on a σ-algebra containing both the compact saturated and the open subsets of X.*

Proof. The case that λ is bounded, i.e., $\lambda_*(X) < \infty$, follows from the preceding results and Theorem 2.4 applied to the algebra of sets $\mathcal{A} = \mathcal{B}$ generated by the lattice $\mathcal{L} = \mathcal{L}^f$ above (with the extended valuation guaranteed by the Smiley–Horn–Tarski theorem) with $\mathcal{K} = \mathcal{P}$ the family of all lenses, which is monocompact (see 5.1).

In the general case, we have already noted that λ_* is locally finite on $\mathcal{O}(X)$. The proof then follows from Lemma 3.3, where one works with a directed family of λ_*-finite open sets whose union is X, the lattice and algebra generated by the both the open sets and the compact saturated sets, and the σ-algebra this algebra generates. (One varifies directly that for each λ_*-finite open set U, the valuation λ_U is outer regular and bounded, hence extends appropriately by the previous paragraph.) We note that the last assertion of Lemma 3.3 guarantees an extension of λ on $\mathcal{Q}(X)$ (since any compact saturated set will eventually be in the directed family of λ_*-finite open sets) and of λ_* on $\mathcal{O}(X)$ (since λ_* is continuous on $\mathcal{O}(X)$ by Lemma 6.2). $\square$

One can also switch the roles of $\mathcal{Q}(X)$ and $\mathcal{O}(X)$ and begin with an *inner regular* valuation $\mu : \mathcal{O}(X) \to [0, \infty]$, a valuation that satisfies: For every $U \in \mathcal{O}$ and every $\gamma < \mu(U)$, there is a $Q \in \mathcal{Q}$ such that $Q \subseteq U$ and $\lambda(V) > \gamma$ for every $V \in \mathcal{O}$ with $Q \subseteq V$. Note that inner regularity implies continuity of the valuation μ. We then consider the *outer extension* μ^* of μ defined for all subsets A by

$$\mu^*(A) := \inf\big\{\mu(U) \colon\ A \subseteq U \in \mathcal{O}(X)\big\}.$$

In the following lemma the nontrivial implication involves a standard compactness argument. Condition (2) provides the appropriate dual to the Wilker condition.

Lemma 6.6. *Let X be a topological space. The following assertions are equivalent*:

(1) *For all $x_1, x_2 \in X$, if U is an open set containing $\uparrow x_1 \cap \uparrow x_2$, then there exists an open set U_1 containing $\uparrow x_1$ and an open set U_2 containing $\uparrow x_2$ such that $U_1 \cap U_2 \subseteq U$.*
(2) *For all compact saturated sets Q_1, Q_2 in X, if U is an open set containing $Q_1 \cap Q_2$, then there exists an open set U_1 containing Q_1 and an open set U_2 containing Q_2. such that $U_1 \cap U_2 \subseteq U$.*

If X is a T_1 space, then condition (1) is easily seen to be equivalent to Hausdorffness and condition (2) then asserts the ability to separate compact sets in a Hausdorff space. We thus refer to spaces satisfying the equivalent conditions as *weakly Hausdorff* spaces.

The following proposition is essentially a dual of Lemmas 6.2 and 6.3, where weak Hausdorffness is the dual condition for the Wilker condition.

Proposition 6.7. *Let X be a coherent sober space that is weakly Hausdorff. For any locally finite inner regular valuation μ on $\mathcal{O}(X)$, the outer extension μ^* is a finite valuation when restricted to the lattice $\mathcal{Q}(X)$ of compact saturated sets. The collection $\mathcal{L}^f$ of subsets A of X satisfying*

$$\sup_{Q \subseteq A} \mu^*(Q) = \inf_{A \subseteq U} \mu(U) < \infty$$

is a lattice containing all compact saturated sets and all μ-finite open sets. Moreover, μ^* is a valuation when restricted to $\mathcal{L}^f$.

Proof. Since μ is locally finite, any compact set Q is covered by finitely many μ-finite open sets. By the modular law it follows that their union is μ-finite, and hence Q is μ^*-finite.

For compact saturated sets Q_1 and Q_2, we have

$$\begin{aligned}
\mu^*(Q_1) + \mu^*(Q_2) &= \inf_{Q_1 \subseteq U_1} \mu(U_1) + \inf_{Q_2 \subseteq U_2} \mu(U_2) && (14)\\
&= \inf_{Q_1 \subseteq U_1, Q_2 \subseteq U_2} \mu(U_1) + \mu(U_2) && (15)\\
&= \inf_{Q_1 \subseteq U_1, Q_2 \subseteq U_2} \mu(U_1 \cup U_2) + \mu(U_1 \cap U_2) && (16)\\
&= \inf_{Q_1 \subseteq U_1, Q_2 \subseteq U_2} \mu(U_1 \cup U_2) + \inf_{Q_1 \subseteq U_1, Q_2 \subseteq U_2} \mu(U_1 \cap U_2) && (17)\\
&= \mu^*(Q_1 \cup Q_2) + \mu^*(Q_1 \cap Q_2), && (18)
\end{aligned}$$

where weak Hausdorffness is used to establish that

$$\mu^*(Q_1 \cap Q_2) = \inf\{\mu(U_1 \cap U_2) \colon\ Q_1 \subseteq U_1,\ Q_2 \subseteq U_2\}$$

and thus ensures the last equality.

In a fashion analogous to Lemma 6.3, we obtain that $\mathcal{L}^f$ is a lattice of sets and that μ^* is a valuation when restricted to $\mathcal{L}^f$. Note that the hypothesis of inner continuity yields that $\mathcal{L}$ contains all μ-finite open sets U. $\square$

It follows from the definition of $\mathcal{L}^f$ that it is approximated from inside by $\mathcal{Q}(X)$ and from outside by $\mathcal{O}(X)$. Hence by Lemma 6.4 the members of the semiring of sets consisting of relative complements $A \setminus B$ for $A, B \in \mathcal{L}^f$ are approximated from inside by lenses. When μ is finite, we can again apply Theorem 2.4 to the algebra $\mathcal{A} = \mathcal{B}$ generated by the lattice $\mathcal{L}^f$ above with $\mathcal{K} = \mathcal{P}$ the monocompact family of all lenses (see 5.1) to obtain the following theorem. The general case then follows from Lemma 3.3 along the same lines as those outlined in the proof of Theorem 6.5.

Theorem 6.8. *Let X be a weakly Hausdorff coherent sober space. Let $\mu : \mathcal{O}(X) \to [0, \infty]$ be a locally finite inner regular valuation on the lattice of open sets. Then μ extends to a σ-additive measure on a σ-algebra containing all open and all compact saturated sets.*

Corollary 6.9. *Let X be a Hausdorff space and let λ be an outer regular valuation on the compact subsets or a locally finite inner regular valuation on the open sets. Then λ extends to a Borel measure.*

Proof. A Hausdorff space is always a coherent sober space. It is clearly weakly Hausdorff and it is a rather straightforward topological exercise to show that it satisfies the Wilker condition (see [19]). Thus the result follows from the previous main theorems of this section. $\square$

7. Coherent spaces and Radon measures

In this section we want to improve the two extension Theorems 6.5 and 6.8 of the previous section. It is our aim to extend a valuation given on the lattice of compact saturated sets or on the lattice of open sets of a coherent sober space X to a Radon measure on a σ-algebra containing the patch topology. For this, we have first to present the appropriate definition of a Radon measure for coherent non-Hausdorff spaces. Recall that the *patch topology* is the coarsest topology containing the given topology on X and the *co-compact topology*, which is generated by the complements of the compact saturated sets in the original topology (see, e.g., [8]).

We adapt methods developed by Topsøe [15–17] and by Berg, Christiansen and Ressel [5, Chapter 2] for Radon measures, mainly on Hausdorff spaces.

Thus let X be a coherent sober space. We denote by $\mathcal{K}$ the collection of all lenses $Q \setminus U$, where Q is compact saturated and U is open in X. As X is coherent, $\mathcal{K}$ is closed under finite intersections and it contains the empty set. Lenses are closed in the patch topology.

Let $\mathcal{K}'$ be the lattice of sets generated by $\mathcal{K}$, that is, the collection of all unions of finitely many lenses; and let $\mathcal{C}$ be the collection of all intersections of nonempty families of members of $\mathcal{K}'$. Except for possibly the whole space X, $\mathcal{C}$ consists of the closed sets of the topology generated by taking the lenses as a subbasis for the closed sets. One might call it the *lens topology*. Clearly the lens topology is contained in the patch topology, but it may be stricty coarser; in fact, the lens topology agrees with the patch topology if and only if X is compact in the original topology.

Lemma 7.1. *The following hold in a coherent sober space X:*

(1) *The collection $\mathcal{C} \cup \{X\}$ forms the closed sets for the lens topology, the smallest topology containing all lenses as closed sets.*
(2) *The lens topology is compact.*
(3) *If A is a closed subset of X, then $A \cap C \in \mathcal{C}$ for each $C \in \mathcal{C}$; in particular, $A \cap C$ is closed in the lens topology.*
(4) *A set A is closed in the patch topology if and only if it is of the form $A \cup C$ for some $C \in \mathcal{C}$ and some A closed in the original topology of X.*
(5) *If $K \in \mathcal{C}$, then K is compact in the patch topology.*

Proof. (1) This follows from the fact that by construction $\mathcal{C} \cup \{X\}$ arises from the closed subbasis $\mathcal{K}$ by first taking finite unions, then arbitrary intersections.

(2) Since X is coherent, it follows that a finite intersection of lenses is again a lens. Thus any family with the finite intersection property will extend (by throwing in all finite intersections) to a filtered family of nonempty lenses. By Lemma 5.1 the intersection is then nonempty. Since the lenses form a subbasis for the lens topology, it follows from the Alexander Subbasis lemma that the space X equipped with the lens topology is compact.

(3) Indeed $A \cap C = C \setminus (X \setminus A)$ is again a lens for each lens C. Since intersection with A distributes over finite unions and intersections of nonempty families, it follows that $A \cap C \in \mathcal{C}$ for each $C \in \mathcal{C}$. The last assertion now follows from part (1).

(4) It follows directly from the definition of the patch topology that its lattice of closed sets contains $\mathcal{K}$ and the closed sets of the original topology. Thus the patch closed sets also contain $\mathcal{C}$. Conversely consider the collection $\mathcal{A}$ of all sets of the form $A \cup C$, where A is closed in X and $C \in \mathcal{C}$. This collection clearly contains $\mathcal{C}$ and the original closed sets (since the empty set belongs to $\mathcal{C}$). The collection $\mathcal{A}$ is easily seen to be closed under finite unions. The fact it is closed under arbitrary intersections follows from the facts that the lattice of closed sets of X is closed under arbitrary intersections, $\mathcal{C}$ is closed under insections of all nonempty families, and part (3):

$$\begin{aligned}\bigcap_{\alpha \in I}(A_\alpha \cup C_\alpha) &= (A_\alpha \cup C_\alpha) \cap \bigcap_{\beta \neq \alpha}(A_\beta \cup C_\beta)\\ &= \bigcap_{\beta \neq \alpha}(A_\alpha \cap A_\beta) \cup \bigcap_{\beta \neq \alpha}\big((A_\alpha \cap C_\beta) \cup (C_\alpha \cap A_\beta) \cup (C_\alpha \cap C_\beta)\big).\end{aligned}$$

(5) It follows from (3) and (4) that the relative patch topology agrees with the relative lens topology on $C \in \mathcal{C}$. Since C is closed in the latter, a compact topology, it follows that C is compact. □

For a Hausdorff space, a Radon measure is defined to be a Borel measure μ with the property that the measure of every compact set is finite and that, for every Borel set B,

$$\mu(B) = \sup\{\mu(K)\colon\ K \text{ compact and } K \subseteq B\}.$$

For coherent spaces it seems appropriate to replace the collection of compact subsets by $\mathcal{C}$, the collection of proper closed sets in the lens topology defined above. By the previous lemma these all remain compact in the patch topology.

Definition 7.2. A *Radon measure* on a coherent sober space is a (countably additive) measure μ defined on a σ-algebra $\mathcal{B}$ containing the patch topology, which is finite on all sets $K \in \mathcal{C}$ and has the property that, for every $B \in \mathcal{B}$,

$$\mu(B) = \sup\{\mu(K)\colon\ K \in \mathcal{C} \text{ and } K \subseteq B\}.$$

If μ is a Radon measure, it is easily seen that the measure of a saturated Borel set A is given by

$$\mu(A) = \sup\{\mu(Q)\colon\ Q \text{ compact saturated and } Q \subseteq A\}.$$

We now can state the main result of this section.

Theorem 7.3. *Let X be a coherent sober space. Suppose that X satisfies the Wilker condition and that λ is an outer regular valuation on the lattice $\mathcal{Q}(X)$ of compact saturated sets or, alternatively, that X is weakly Hausdorff and λ a locally finite inner regular valuation on the lattice $\mathcal{O}(X)$ of open sets. Then there is a unique Radon measure μ extending λ, defined on the σ-algebra generated by the patch topology.*

For the proof we need the notion of tightness: If $\mathcal{L}$ is a lattice of sets containing the empty set, a set function $\lambda : \mathcal{L} \to [0, +\infty[$ is called *tight*, if $\lambda(\emptyset) = 0$ and for arbitrary $L, L' \in \mathcal{L}$ with $L' \subseteq L$:

$$\lambda(L) - \lambda(L') = \sup\{\lambda(L_1)\colon\ L_1 \in \mathcal{L},\ L_1 \subseteq L \setminus L'\}.$$

The following lemma is due to Topsøe [16], who uses the term *τ-smooth* instead of *continuous*.

Lemma 7.4. *Let $\mathcal{L}$ be a lattice of sets and let $\lambda : \mathcal{L} \to [0, +\infty[$ be a tight set function. Then the following properties hold*:

(i) *λ is a valuation on $\mathcal{L}$.*

(ii) *If $\mathcal{L}$ is compact* (*that is, every nonempty subcollection of $\mathcal{L}$ with the finite intersection property has nonempty intersection*), *then λ is continuous in the sense that whenever (L_α) is a filtered family in $\mathcal{L}$ whose intersection $L := \bigcap_\alpha L_\alpha$ belongs to $\mathcal{L}$, then $\lambda(L) = \inf_\alpha \lambda(L_\alpha)$.*

Proof. (i) It is clear that λ is monotone. As $(A \cup B) \setminus B = A \setminus (A \cap B)$, tightness implies that $\lambda(A \cup B) - \lambda(B) = \lambda(A) - \lambda(A \cap B)$, which is modularity.

(ii) Suppose by way of contradiction that $\delta = \inf_\alpha \lambda(L_\alpha) - \lambda(L) > 0$. For any fixed α_0, we may find, by the tightness hypothesis, a member L_1 of $\mathcal{L}$ contained in $L_{\alpha_0} \setminus L$ such that

$$\lambda(L_{\alpha_0}) - \lambda(L) - \lambda(L_1) < \delta.$$

As $\bigcap_\alpha L_1 \cap L_\alpha = L_1 \cap L = \emptyset$, by compactness there is an α_1 such that $L_1 \cap L_{\alpha_1} = \emptyset$. As the family (L_α) is filtered, there is an index $\alpha \geqslant \alpha_0, \alpha_1$ such that $L_1 \cap L_\alpha = \emptyset$. Thus

$$\begin{aligned} 0 &\leqslant \lambda(L_{\alpha_0}) - \lambda(L_\alpha) - \lambda(L_1) \\ &\leqslant \lambda(L_{\alpha_0}) - \lambda(L) - \lambda(L_1) \\ &< \delta. \end{aligned}$$

From these inequalities we conclude that $\lambda(L_\alpha) - \lambda(L) < \delta$, which is in contradiction to the choice of δ. □

We may apply the preceding lemma in particular to the lattice $\mathcal{K}'$ of finite unions of lenses in X, as this lattice is compact by 7.1.

We now go back to the hypotheses of the preceding section and suppose that X is a coherent sober space satisfying the Wilker condition or, alternatively, is weakly Hausdorff. Given an outer regular valuation on $\mathcal{Q}(X)$ or, alternatively, a locally finite inner regular valuation on $\mathcal{O}(X)$, we have seen (Theorems 6.5 and 6.8) that there is a σ-algebra $\mathcal{A}$ of subsets of X containing all open and all compact saturated sets and a countably additive extension $\bar{\lambda}$ of λ to $\mathcal{A}$. Note that $\mathcal{K}' \subseteq \mathcal{A}$. We now can prove:

Lemma 7.5. *The restriction of $\bar{\lambda}$ to the collection $\mathcal{K}'$ of finite unions of lenses is tight.*

Proof. Let L and L' be finite unions of lenses with $L' \subseteq L$ and let $\varepsilon > 0$. It follows from Lemma 6.4 that $L \setminus L'$ belongs the ring of subsets generated by $\mathcal{L}^f$, the lattice of Lemma 6.3; thus it can be written as the union of pairwise disjoint sets $L_i \setminus M_i$, $i = 1, \dots, n$, with $L_i, M_i \in \mathcal{L}^f$. By Lemma 6.4, for every i there is a lens $Q_i \setminus U_i$ contained in $L_i \setminus M_i$ such that $\bar{\lambda}(L_i \setminus M_i) < \bar{\lambda}(Q_i \setminus U_i) + \frac{\varepsilon}{n}$. For the union L_1 of the pairwise disjoint lenses $Q_i \setminus U_i$ we then have $\bar{\lambda}(L) - \bar{\lambda}(L') < \bar{\lambda}(L_1) + \varepsilon$. □

We now extend the set function $\bar{\lambda}$ from $\mathcal{K}'$ to $\mathcal{C}$ by defining for all $K \in \mathcal{C}$:

$$\mu(K) = \inf\{\bar{\lambda}(L'):\ K \subseteq L' \in \mathcal{K}'\}. \tag{19}$$

The next lemma is taken from Topsøe [15, Part I, Section 5], but our proof is different. Note that the continuity hypothesis is fulfilled by Lemma 7.4, as $\mathcal{K}'$ is compact.

Lemma 7.6. *The outer extension μ is tight on $\mathcal{C}$.*

Proof. The proof is done in several steps. We first show:

(i) If (L'_α) is any filtered family in $\mathcal{K}'$ and $K = \bigcap_\alpha L'_\alpha \in \mathcal{C}$, then $\mu(K) = \inf_\alpha \bar{\lambda}(L'_\alpha)$: Indeed, for every $L' \in \mathcal{K}'$ with $K \subseteq L'$, the family $(L' \cup L'_\alpha)_\alpha$ is filtered in $\mathcal{K}'$ and has intersection L', whence $\inf_\alpha \bar{\lambda}(L' \cup L'_\alpha) = \bar{\lambda}(L')$ by the continuity of $\bar{\lambda}$ on $\mathcal{K}'$. Hence,

$\inf_\alpha \bar{\lambda}(L'_\alpha) \leqslant \inf_\alpha \bar{\lambda}(L' \cap L'_\alpha) = \bar{\lambda}(L')$. As, by definition, $\mu(K)$ is the infimum of the $\bar{\lambda}(L')$, where L' runs through all members of $\mathcal{K}'$ containing K, we have the desired result.

(ii) μ is a valuation on $\mathcal{C}$: It is clear that μ is strict and monotone. For modularity, we use that $\bar{\lambda}$ is modular on $\mathcal{K}'$, and we take arbitrary sets $K, K' \in \mathcal{C}$. We choose filtered sets $(L'_\alpha)_\alpha$ and $(M'_\beta)_\beta$ in $\mathcal{K}'$ with intersection K and K', respectively. Then $(L'_\alpha \cap M'_\beta)_{\alpha,\beta}$ and $(L'_\alpha \cup M'_\beta)_{\alpha,\beta}$ are filtered families, intersecting in $K \cap K'$ and $K \cup K'$, respectively. Using (i), we then have

$$\begin{aligned}
\mu(K) + \mu(K') &= \inf_\alpha \bar{\lambda}(L'_\alpha) + \inf_\beta \bar{\lambda}(M'_\beta) \\
&= \inf_{\alpha,\beta} \bar{\lambda}(L'_\alpha) + \bar{\lambda}(M'_\beta) \\
&= \inf_{\alpha,\beta} \bar{\lambda}(L'_\alpha \cap M'_\beta) + \bar{\lambda}(L'_\alpha \cup M'_\beta) \\
&= \mu(K \cap K') + \mu(K \cup K').
\end{aligned}$$

(iii) In order to prove tightness, we take $K, K' \in \mathcal{C}$ with $K' \subseteq K$. By the definition of μ, we may find $L', M' \in \mathcal{K}'$ such that $K \subseteq L'$, $K' \subseteq M' \subseteq L'$ and

$$\bar{\lambda}(L') - \mu(K) < \varepsilon, \qquad \bar{\lambda}(M') - \mu(K') < \varepsilon.$$

As $\bar{\lambda}$ is tight on $\mathcal{K}'$, we can find a $L'_1 \in \mathcal{K}'$ such that $L'_1 \subseteq L' \setminus M'$ and

$$\bar{\lambda}(L') - \bar{\lambda}(M') - \bar{\lambda}(L'_1) < \varepsilon.$$

We now set $K_1 = L'_1 \cap K$. Then $K_1 \subseteq K \setminus K'$. Using (ii), we obtain

$$\begin{aligned}
\bar{\lambda}(L'_1) - \mu(K_1) &= \bar{\lambda}(L'_1) - \mu(K \cap L'_1) \\
&= \mu(L'_1) - \mu(K) - \mu(L'_1) + \mu(K \cap L'_1) \\
&\leqslant -\mu(K) + \mu(L') < \varepsilon.
\end{aligned}$$

From the above inequalities we deduce the desired inequality:

$$\begin{aligned}
\mu(K) - \mu(K') - \mu(K_1) &= \lambda'(L') - \bar{\lambda}(M') - \bar{\lambda}(L'_1) + (\mu(K) - \bar{\lambda}(L')) \\
&\quad + (\bar{\lambda}(M') - \mu(K')) + (\bar{\lambda}(L'_1) - \mu(K_1)) \\
&\leqslant |\bar{\lambda}(L') - \bar{\lambda}(M') - \bar{\lambda}(L'_1)| + |\mu(K) - \bar{\lambda}(L')| \\
&\quad + |\bar{\lambda}(M') - \mu(K')| + |\bar{\lambda}(L'_1) - \mu(K_1)| \\
&< 4\varepsilon. \qquad \square
\end{aligned}$$

We are now ready to complete the proof of Theorem 7.3.

Proof. We define μ on the lattice $\mathcal{C}$ by Eq. (19). Then μ and $\bar{\lambda}$ restricted to the compact saturated sets clearly agree.

By Lemmas 7.6 and 7.4 μ is tight and a valuation on $\mathcal{C}$. In the setting of Theorem 2.4, let $\mathcal{K} = \mathcal{P} = \mathcal{C}$, a compact, hence monocompact family, let $\mathcal{B}$ be the semiring $\{C_1 \setminus C_2\colon C_1, C_2 \in \mathcal{C}\}$, and let $\mathcal{A}$ be the algebra generated by $\mathcal{C}$. By the Smiley–Horn–Tarski theorem, extend μ to a finitely additive measure on $\mathcal{A}$. It is a straightforward argument from tightness that $\mathcal{P}^f = \mathcal{C}^f$ approximates $\mathcal{B}$ in the sense of Theorem 2.4 (modify the proof of Lemma 7.5). Thus by Theorem 2.4, μ restricted to $\mathcal{B}$ extends to a measure on the σ-algebra $\mathcal{A}_\sigma(\mathcal{B})$ generated by $\mathcal{B}$. Call this extension ν.

For each compact saturated set K, we consider the measure ν_K on $\mathcal{A}_\sigma(\mathcal{B})$ defined by $\nu_K(A) = \nu(A \cap K)$. As a consequence of Lemma 3.3 (with the collection of U_j being the compact saturated sets) and the remark following it, we conclude that ν extends to a measure on the σ-algebra $\mathcal{E}$ of all sets A such that $A \cap K \in \mathcal{A}_\sigma(\mathcal{B})$ for all compact saturated sets K. This includes, in particular, all patch closed sets (see Lemma 7.1(3)).

We consider the collection $\mathcal{F} \subseteq \mathcal{E}$ consisting of all $F \in \mathcal{E}$ satisfying

$$\nu(F) = \sup\bigl\{\mu(C) \colon C \in \mathcal{C},\ C \subseteq F\bigr\}.$$

We note that $\mathcal{F}$ contains all closed sets A of the original topology, since by definition of ν, $\nu(A)$ is the supremum of all $\mu(A \cap C)$, $C \in \mathcal{C}$ (see the proof of Lemma 3.3), and each $A \cap C \in \mathcal{C}$ (Lemma 7.1(3)). Thus, in particular, $X \in \mathcal{F}$. It follows readily from the tightness of μ on $\mathcal{C}$ (Lemma 7.6) that $\mathcal{F}$ is closed under the operation of relative complementation, and hence under complementation, since X is closed and hence in $\mathcal{F}$. It is easily verified (from the countable additivity of ν) to be closed under countable disjoint unions. It thus follows that $\mathcal{F}$ is a σ-algebra containing $\mathcal{C}$ and all closed sets; by Lemma 7.1(4) it contains all patch closed sets, hence all patch open sets. Thus ν restricted to $\mathcal{F}$ is a Radon measure on a σ-algebra containing the patch topology. Clearly a Radon measure is uniquely determined by its values on $\mathcal{C}$. □

There is also in the literature an alternative approach that one can adopt, namely we can apply Theorem 2(ii) in Topsøe [17]. Consider the collection $\mathcal{B}$ of all A which satisfy

$$\mu_*(K) = \mu_*(K \cap A) + \mu_*(K \setminus A) \quad \text{for all } K \in \mathcal{C}.$$

Then $\mathcal{B}$ is a σ-algebra and the restriction of the inner measure μ_* to $\mathcal{B}$ is a measure. The σ-algebra $\mathcal{B}$ contains all the subsets $C \subseteq X$ with the property that $C \cap K \in \mathcal{C}$ for all $K \in \mathcal{C}$. In particular, $\mathcal{B}$ contains $\mathcal{C}$, hence also all compact saturated sets, and all open subsets of X. In fact, $\mathcal{B}$ is the greatest σ-algebra to which μ can be extended as a $\mathcal{C}$-inner regular measure.

The proof of this result can be copied from Topsøe's paper [16] (see Theorem 1 and the lemma preceding it), or likewise from the proof of Theorem 1.4, Chapter 2, in [5]. Although in these papers one only deals with Hausdorff spaces, the proof remains valid for our more general situation. This finishes the proof of Theorem 7.3.

Remark. The extension to Radon measures of outer continuous valuations on the lattice of compact saturated subsets on a coherent sober space satisfying the Wilker condition has been investigated by Norberg and Vervaat [14]. Our Lemmas 6.2 and 6.3 are due to them. But they define a Radon measure to be a measure defined on the σ-algebra $\mathcal{B}$ generated by the open and the compact saturated sets which is inner regular with respect to the family $\mathcal{K}'$ of finite unions of lenses, that is $\mu_*(B)$ is the sup of the $\bar{\lambda}(L')$ where L' is a finite union of lenses contained in B. They claim in their Theorem 3.7 that they can extend valuations as above to Radon measures in their sense.

This result seems to be wrong. If we consider the unit interval $X = [0, 1]$ with the Scott topology, the compact saturated sets are the intervals $[a, 1]$. Choosing the valuation

$\lambda([a,1]) = 1 - a$, its extension will be Lebesgue measure. But this is not inner regular with respect to finite unions of lenses. The lenses are the intervals $[a,b]$, but there are Borel sets of positive measure in the unit interval that do not contain any nonsingleton interval.

The error in the proof of Theorem 3.7 in [14] has its origin in an incomplete citation of Exercise I-6-1 in Neveu's book [12].

We had to modify the notion of a Radon measure by considering the collection $\mathcal{C}$ of intersections of finite unions of lenses in order to arrive at the desired result.

8. Stably locally compact spaces

One of the notable achievements in the study of T_0-spaces has been the identification of the important classes of stably compact and stably locally compact spaces and various characterizations of them, see, for example, [8, Chapter VI-6]. We recall that a space is *stably locally compact* if it is a locally compact coherent sober space.

Lemma 8.1. *Let X be a stably locally compact space. Then X is weakly Hausdorff and satisfies the Wilker condition.*

Proof. Let $x, y \in X$ and let U be an open set containing $\uparrow x \cap \uparrow y$. By local compactness and coherence, the collection

$$\{K \cap L\colon\ x \in \operatorname{int}(K),\ y \in \operatorname{int}(L),\ K, L \text{ are compact saturated}\}$$

is a filtered family of compact saturated sets with intersection $\uparrow x \cap \uparrow y$. It follows from the Hofmann–Mislove machinery [8, Theorem II-1.21] that $K \cap L \subseteq U$ for some compact K, L with $x \in \operatorname{int}(K)$, $y \in \operatorname{int}(L)$. Thus X is weakly Hausdorff.

Let Q be a compact saturated set and let U_1, U_2 be open sets with $Q \subseteq U_1 \cup U_2$. For each $x \in U_1$, pick a compact saturated neighborhood Q_x of x such that $Q_x \subseteq U$ (this is always possible in any locally compact space). Similarly for each $y \in U_2$, pick a compact saturated neighborhood Q_y such that $Q_y \subseteq U_2$. Then finitely many of the $\{Q_x\colon\ x \in U_1\} \cup \{Q_y\colon\ y \in U_2\}$ must cover Q. Set Q_1 equal to the union of those sets in that finite cover that lie in U_1 and Q_2 equal to the union of the ones lying in U_2. Then $Q \subseteq Q_1 \cup Q_2$ as needed for the Wilker condition. □

Lemma 8.2. *Let X be a locally compact sober space. Then a valuation on the open sets is continuous if and only if it is inner regular. If X is additionally coherent, then any finite-valued valuation on the compact saturated sets is outer regular if and only if it is continuous* (*with respect to filtered intersections*).

Proof. We remarked in the preceding section that an inner regular valuation on the open sets is always continuous. Conversely, consider a continuous valuation μ. Note that any open set U is the directed union of open sets V such that there exists a compact saturated set Q with $V \subseteq Q \subseteq U$. By continuity of μ, for any $\gamma < \mu(U)$, we may pick an open set V and a compact set Q such that $V \subseteq Q \subseteq U$ and $\gamma < \mu(V)$. It then follows easily that μ is inner regular.

Suppose that λ is a continuous valuation on $Q(X)$, the lattice of compact saturated sets. Let Q be a compact saturated set. Since X is locally compact, the compact saturated neighborhoods of Q form a filtered or descending family with intersection Q. By continuity for any $\varepsilon > 0$, there exists a compact neighborhood A of Q such that $\lambda(A) < \lambda(Q) + \varepsilon$. Choose the U in the definition of outer continuity to be the interior of A. The other direction follows from the Hofmann–Mislove machinery [8, Theorem II-1.21], since any filtered intersection of compact saturated sets is eventually contained in any neighborhood of the intersection. □

The next theorem follows from Theorem 7.3 of the previous section, in light of the preceding lemmas.

Theorem 8.3. *Let X be a stably locally compact space. Further, let λ be a locally finite continuous valuation on the lattice of open sets or, alternatively, a finite-valued continuous valuation on the lattice of compact saturated sets. Then λ can be extended to a Radon measure on the σ-algebra generated by the patch topology.*

9. Future directions and open problems

We have relied heavily on the property of coherence for obtaining Borel measures on the σ-algebra generated by both the compact saturated and the open sets. Are there more general results? In particular, does a locally finite continuous valuation on a locally compact sober space extend to a Borel measure of this type?

Theorem 8.3 seems to characterize Radon measures on stably locally compact spaces. But Theorem 7.3 does not characterize Radon measures on coherent sober spaces. There is still the problem of characterizing those valuations on the set of compact saturated subsets of a coherent sober spaces that extend to Radon measures. In the cases that we are dealing with one may ask the question whether our Radon measures are even outer regular, i.e., whether the measure of a Borel set is also the infimum of the measures of the patch-open sets containing it.

In regard to Section 4, we recall a problem of Reinhold Heckmann. A positive solution would generalize the results of that section. Call a valuation μ on the open sets of a topological space *point continuous* if for any U open and $\varepsilon > 0$, there exists a finite subset F of U such that $\mu(U) - \varepsilon < \mu(V)$ for all open sets V containing F. Does a point continuous valuation on a monotone convergence space (or sober space) extend to a measure?

References

[1] S. Abramsky, A. Jung, Domain theory, in: S. Abramsky, et al. (Eds.), Handbook of Logic in Computer Science, vol. 3, Clarendon Press, Oxford, 1995.

[2] M. Alvarez-Manilla, Measure theoretic results for continuous valuations on partially ordered spaces, Dissertation, Imperial College, London, 2000.

[3] M. Alvarez-Manilla, Extension of valuations on locally compact sober spaces, Topology Appl. 124 (2002) 397–433.

[4] M. Alvarez-Manilla, A. Edalat, N. Saheb-Djahromi, An extension result for continuous valuations, J. London Math. Soc. (2) 61 (2000) 629–640.
[5] C. Berg, J.P.R. Christensen, P. Ressel, Harmonic Analysis on Semigroups. Theory of Positive Definite and Related Functions, Graduate Texts in Mathematics, vol. 100, Springer, Berlin, 1984, 289 pp.
[6] P. Billingsley, Probability and Measure, second ed., Wiley, New York, 1986.
[7] C. Dellacherie, P.-A. Meyer, Probabilities and Potential, North-Holland, Amsterdam, 1975.
[8] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, Continuous Lattices and Domains, Cambridge University Press, Cambridge, 2003.
[9] C. Jones, Probabilistic non-determinism, Ph.D. thesis, University of Edinburgh, 1989.
[10] K. Keimel, R. Tix, Valuations and measures, Technische Universität Darmstadt, 1996, unpublished manuscript.
[11] J. Lawson, Valuations on continuous lattices, in: R.-E. Hofmann (Ed.), Math. Arbeitspapiere, vol. 27, University of Bremen, 1982, pp. 204–225.
[12] J. Neveu, Bases Mathématiques du Calcul des Probabilités, Masson & Cie, Paris, 1964.
[13] T. Norberg, Existence theorems for measures on continuous posets, with applications to random set theory, Math. Scand. 64 (1989) 15–51.
[14] T. Norberg, W. Vervaat, Capacities on non-Hausdorff spaces, in: W. Vervaat, H. Holwerda (Eds.), Probability and Lattices, in: CWI Tract, vol. 110, CWI, Amsterdam, 1997, pp. 133–150.
[15] F. Topsøe, Topology and Measure, Lecture Notes in Mathematics, vol. 133, Springer, Berlin, 1970, xiv+79pp.
[16] F. Topsøe, Compactness in spaces of measures, Studia Math. 36 (1970) 195–212.
[17] F. Topsøe, On construction of measures, in: Proceedings of the Conference "Topology and Measures I", Zinnowitz 1974, Greifswald University, 1978, pp. 343–381.
[18] G. Weidner, Bewertungen auf stetigen Verbänden, Master's thesis, Technische Universität Darmstadt, 1995.
[19] P. Wilker, Adjoint product and hom functors in general topology, Pacific J. Math. 34 (1970) 269–283.